Integrals in Economics - Surplus

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- The term "surplus" is used to denote gains from trade created by a seller producing a commodity and allocating it to a consumer who values it.

- For consumers, the **demand curve**, $p = D(q)$, relates the unit price p of a commodity to its quantity demanded q. It is assumed that $D(q)$ is **decreasing**. If $\bar{q}$ units are traded at price $\bar{p}$ in the market, consumers who are willing to pay a price higher than $\bar{p}$ experience gains from trade. The difference between what consumers are willing to pay for $\bar{q}$ units and what they actually pay for them is called the **consumers' surplus**. Mathematically, the consumer's surplus is defined as

$$CS = \int_0^{\bar{q}} D(q)\, dq - \bar{p}\bar{q}.$$

  Geometrically, the consumer's surplus is the area of region bounded above by the demand curve and below by the line $p = \bar{p}$, ranging from $q = 0$ to $q = \bar{q}$.

- For producers, the **supply curve**, $p = S(q)$, relates the unit price of a commodity to the quantity supplied q (that producers are willing to sell). It is assumed that $S(q)$ is **increasing**. If $\bar{q}$ units are traded in the market at price $\bar{p}$, those producers who are willing to supply the commodity at a lower price experience gains from trade. The difference between what producers actually receive for $\bar{q}$ units and what they are willing to accept is called the **producers' surplus**. The producers' surplus is defined as

$$PS = \bar{p}\bar{q} - \int_0^{\bar{q}} S(q)\, dq.$$

  The producers' surplus represents the area of region bounded above by the line $p = \bar{p}$ and below by the supply curve, ranging from $q = 0$ to $q = \bar{q}$.

- Gains from trading $\bar{q}$ units of a commodity is called the **total surplus**. It is defined as the area of region between the supply and demand curve, ranging from $q = 0$ to $q = \bar{q}$,

$$TS(\bar{q}) = \int_0^{\bar{q}} [D(q) - S(q)]\, dq.$$

  As you can see, the total surplus is the sum of the consumers' surplus and producers' surplus, $TS = CS + PS$.

Example: Suppose the demand curve is $D(q) = 2^{(6-\frac{q}{18})}$ and the supply curve is $S(q) = 4^{(1+\frac{q}{18})}$.

1. Compute the consumers' surplus when $q = 18$ and $p = 32$. How much dose the consumers' surplus increases when the quantity of trade changes to 21 and the price changes to $2^{\frac{29}{6}}$?

For $q=18$, $P=32$, $CS(18) = \int_0^{18} 2^{(6-\frac{q}{18})}\, dq - 18\times 32$

$= \int_6^5 2^u (-18)\, du - 2^6\times 3^2 = 18\int_5^6 2^u\, du - 2^6\times 3^2 = \frac{18}{\ln 2} 2^u \Big|_5^6 - 2^6\times 3^2$

$\begin{cases} u = 6 - \frac{q}{18} \\ du = -\frac{1}{18} dq \end{cases}$

$= 2^6\times 3^2 \left(\frac{1}{\ln 2} - 1\right) \approx 254.99$

Similarly, for $q=21$, $P=2^{\frac{29}{6}}$, $CS(21) = \int_0^{21} 2^{(6-\frac{q}{18})}\, dq - 21\times 2^{\frac{29}{6}} = \frac{18}{\ln 2}\left(2^6 - 2^{\frac{29}{6}}\right) - 21\times 2^{\frac{29}{6}}$

≈ 322.97

$CS(21) - CS(18) \approx 67.98$

2. Find the value q^* such that $D(q^*) = S(q^*)$. Show that the total surplus $TS(q)$ obtains maximum at $q = q^*$.

1. $D(q^*) = S(q^*) \Rightarrow 2^{(6-\frac{q^*}{18})} = 4^{(1+\frac{q^*}{18})} = 2^{2+\frac{q^*}{9}} \Rightarrow 6-\frac{q^*}{18} = 2+\frac{q^*}{9} \Rightarrow q^* = 24$

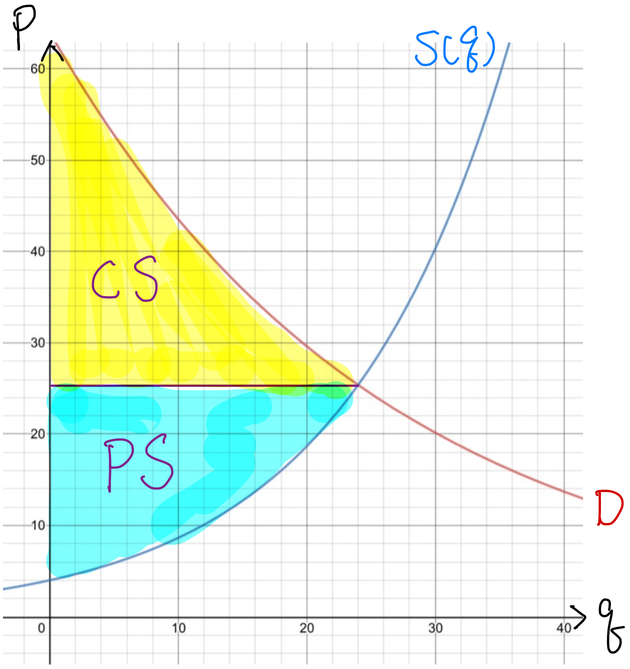
2. $\frac{d}{dq} TS(q) = \frac{d}{dq} \int_0^q D(t) - S(t)\, dt = D(q) - S(q)$. Thus $\frac{d}{dq} TS = 0$ for $q = q^*$.

$\because$ $D(q)$ is decreasing and $S(q)$ is increasing $\therefore$ $D(q) - S(q)$ is decreasing.

Hence $\frac{d}{dq} TS > 0$ for $0 < q < q^*$, and $\frac{d}{dq} TS < 0$ for $q > q^*$. Thus TS obtains maxi at $q = q^*$.

3. Calculate consumers' surplus, producers' surplus, and total surplus when $q = q^*$, $p = D(q^*) = S(q^*)$. Draw graphs of $D(q)$ and $S(q)$ and indicate regions which represent these surpluses.

When $q = q^* = 24$, $P = D(q^*) = S(q^*) = 2^{\frac{14}{3}}$.



$CS(24) = \int_0^{24} 2^{(6-\frac{q}{18})} dq - 2^{\frac{14}{3}} \times 24$

$= \frac{18}{\ln 2}(2^6 - 2^{\frac{14}{3}}) - 2^{\frac{14}{3}} \times 24.$

$PS(24) = 2^{\frac{14}{3}} \times 24 - \int_0^{24} 4^{(1+\frac{q}{18})} dq = 2^{\frac{14}{3}} \times 24 - \frac{9}{\ln 2}(2^{\frac{14}{3}} - 2^2)$

$TS(24) = CS(24) + PS(24) = \frac{9}{\ln 2}(2^7 + 2^2 - 3 \times 2^{\frac{14}{3}})$

4. Now due to improvements of technology, the supply curve shifts down to $S(q) = 4^{(\frac{1}{2}+\frac{q}{18})}$. Find the maximum total surplus for this case.

Solve q^* such that $D(q^*) = S(q^*)$

$\Rightarrow 2^{(6-\frac{q^*}{18})} = 4^{(\frac{1}{2}+\frac{q^*}{18})} \Rightarrow 6 - \frac{q^*}{18} = 1 + \frac{q^*}{9} \Rightarrow q^* = 30.$

Total surplus obtains maximum value when $q^* = 30$.

$TS(30) = \int_0^{30} 2^{(6-\frac{q}{18})} - 4^{(\frac{1}{2}+\frac{q}{18})} dq = \frac{9}{\ln 2}\left[2^7 - 2^{\frac{16}{3}} - 2^{\frac{10}{3}} + 2\right]$

In general, the demand function $p = D(q)$ is decreasing with inverse function $q = D^{-1}(p)$.

For price p, the consumer surplus at price p is

$$CS(p) \equiv \int_0^{D^{-1}(p)} D(q)\,dq - p \cdot D^{-1}(p).$$

Let's compute $\frac{d}{dp} CS(p)$.

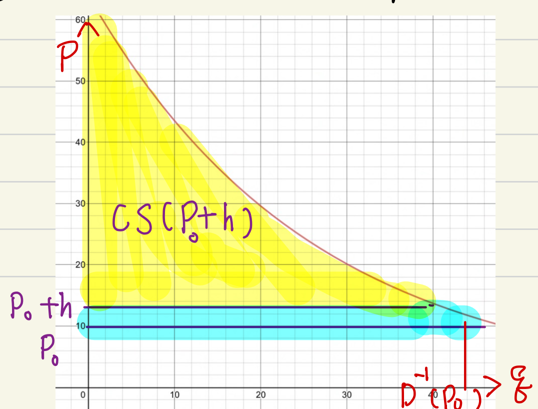
By the fundamental theorem of Calculus

$$\frac{d}{dp} CS(p) = \frac{d}{dp}\left[\int_0^{D^{-1}(p)} D(q)\,dq - p \cdot D^{-1}(p)\right]$$

$$= D(D^{-1}(p)) \cdot \frac{d}{dp} D^{-1}(p) - D^{-1}(p) - p \frac{d}{dp} D^{-1}(p)$$

$$= p \cdot \frac{d}{dp} D^{-1}(p) - D^{-1}(p) - p \frac{d}{dp} D^{-1}(p)$$

$= -D^{-1}(p) < 0$ because $D^{-1}(p)$ should be the corresponding q which is positive. Hence $CS(p)$ is decreasing with respect to p.



As p increases, $CS(p)$ decreases.

And $\frac{d}{dp} CS(p) = -D^{-1}(p)$.